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The dynamic storage capacity of a periodically heated slab

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Abstract

The well-known problem of evaluating the dynamic heat storage capacity of a 1D slab is analysed extending the results to the more general case of periodic excitation profiles of any form. Equations in time and frequency domain to calculate the storage capacity are obtained and applied to some cases, showing that harmonic heating is not the most efficient way to store energy in finite and semi-infinite slabs. Quantitative comparisons show that, for slab of finite thickness, intermittent heating may yield up to 40% more heat storage capacity (and up to 52% for semi-infinite slab) than harmonic heating.

Keywords: Heat storage; 1-D slab; Periodic conduction; Dynamic head capacity

1. Introduction

The dynamic storage capacity of a heated stab is a typical conduction problem with important practical applications, for example in conjunction with the energy saving efforts for passive buildings [1,2] or laser heating of materials [3], etc. As pointed out by Magyari and Keller [4], the heat storage capacity of a slab excited harmonically to one end and insulated to the other, shows, as a function of its thickness, a curve that reaches a maximum and then it tends to the asymptotic value of the semi-infinite medium in an oscillating manner. In [4] the first and subsequent relative maxima where carefully analysed (also for a non-Fourier conduction equation) and connected to the coherent superposition of two basic "thermal waves" propagating in opposite directions. The existence of the above mentioned absolute maximum shows that an optimum slab size exists (depending on the frequency of the exciting input) that maximises the energy storage. However, harmonic heating is not a common case in practical situations, whereas periodic heating of arbitrary shape is certainly more common. This rises the question whether a different heating profile can increase the heating storage and how the above mentioned optimum slab thickness may depend on the heating profile. The scope of the present work is to analyse the more general case of periodic (not necessarily harmonic) heating, and to find a way to compare the efficiency of different heating modalities. To this aim, equations in frequency and time domain are developed and used to evaluate the instantaneous energy content and the maximum energy storage for generally periodic heating. In the following analysis it is assumed that "steady" periodic regime is always reached (i.e. memory of the initial state is lost and all non-periodic transient phenomena are absent). Moreover, only the simplest case of imposed periodic temperature to a slab end and adiabatic condition on the other end are considered here, as other kind of conditions (convection, imposed periodic flux etc.) can be introduced using the substantially same method proposed here.

2. Basic equations

Starting with the energy conservation equation for a homogeneous incompressible 1D slab:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

define the temperature Fourier transform as:

$$T(t,x) = T_a(x) + \int_{-\infty}^{+\infty} S(x,\omega) e^{i\omega t}\, \mathrm{d}\omega$$

where T_a is the time-average temperature that satisfies the steady version of Eq. (1) and, for the present case, it is a con-

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Nomenclature

c	specific heat
C	dynamic capacity in frequency domain
d	slab thickness
h	function
k	thermal conductivity
K	dynamic capacity in time domain
m	integer number
q	heat flux
Q	heat storage
r	function
S	temperature Fourier transform
t	time
T	temperature
$\mathcal{T}$	period
U	internal energy
u	internal energy fluctuation
x	coordinate

Greek symbols

α	thermal diffusivity: $\frac{k}{\rho c}$
β	constant
δ	Dirac-delta function
ε	effusivity
η	square of non-dimensional slab thickness: $\eta = \xi^2$
ξ	non-dimensional slab thickness
τ, σ, μ, χ	non-dimensional times
κ, Ψ, Λ	non-dimensional functions
ρ	density
ω	frequency

Indexes

a	average
d	slab of finite thickness
h	harmonic
i	intermittent
0	slab surface at $x = 0$
∞	semi-infinite wall

stant or a linear function of x. The following ordinary differential equation is then found (where apex means derivation respect to x):

$$i\omega S(x,\omega) = \alpha S''(x,\omega)$$

The general solution is:

$$S(x,\omega) = S_+(\omega)e^{\beta x} + S_-(\omega)e^{-\beta x}$$

with $\beta = \sqrt{\frac{i\omega}{\alpha}}$, and $S_\pm(\omega)$ are arbitrary functions of ω. Consider now the following boundary conditions: periodic excitation (not necessarily harmonic) at one boundary ($x = 0$) and adiabatic insulation at the other one ($x = d$):

$$T(0,t) = T_0(t); \qquad q(d,t) = 0$$

where $T_0(t)$ is an arbitrary periodic function (with period $\mathcal{T}$). These conditions extend to arbitrary periodic excitation those studied in [4]. It is then easy to see that $T_a(x) = T_a$ and that:

$$S(0,\omega) = S_0(\omega); \qquad S'(d,\omega) = 0$$

where $S_0(\omega)$ is defined by the equation $T_0(t) = T_a(0) + \int_{-\infty}^{+\infty} S_0(\omega)e^{i\omega t}\,\mathrm{d}\omega$. Then, the solution of the differential problem is:

$$S(x,\omega) = S_0(\omega)\frac{\cosh(\beta(x-d))}{\cosh(\beta d)} \tag{2}$$

The particular case of a semi-infinite slab will be also analysed and, in this case, the solution of the related problem is simply:

$$S_\infty(x,\omega) = S_0(\omega)e^{-\beta x}$$

The total internal energy stored into one square meter of slab at time t can be evaluated, for a finite slab, through the integral:

$$U_d(t) = U_{d,a} + u_d(t) = \rho c\int_0^d T_a(x)\,\mathrm{d}x + \rho c\int_0^d T'(x,t)\,\mathrm{d}x$$

where $T'(x,t) = T(x,t) - T_a(x)$ and $u_d(t)$ is the internal energy fluctuation around the time average $U_{d,a}$. Using the result (2) we obtain:

$$u_d(t) = \rho c\int_{-\infty}^{+\infty} S_0(\omega)\frac{\tanh(\beta d)}{\beta}e^{i\omega t}\,\mathrm{d}\omega \tag{3}$$

For a semi-infinite slab ($d \to \infty$) the same calculation yields:

$$u_\infty(t) = \rho c\int_{-\infty}^{+\infty} S_0(\omega)\frac{1}{\beta}e^{i\omega t}\,\mathrm{d}\omega \tag{4}$$

3. The dynamic heat capacity

There is a need to define the dynamic heat capacity of a 1D slab in a general sense. The definition of the "static" heat capacity of a solid incompressible body can be written as:

$$C_s = \frac{\mathrm{d}Q}{\mathrm{d}T}$$

where the ratio is taken over a quasi-static process. For a dynamic process, the ratio $\frac{\dot{Q}}{\dot{T}} = \frac{\mathrm{d}Q}{\mathrm{d}T}$ has no longer a univocal meaning, as the body temperature is not univocally defined. Consider now the 1D slab harmonically heated, we can then set: $T_0(t) = T_0e^{i\omega t}$ and from Eq. (3) one can write:

$$u(t) = \rho cd\frac{\tanh(\beta d)}{\beta d}T_0e^{i\omega t} = \rho c\frac{\tanh(\beta d)}{\beta}T_0(t)$$

thus, the heat flux at $x = 0$ can be written as:

$$\dot{Q} = \dot{u} = \rho c\frac{\tanh(\beta d)}{\beta}\dot{T}_0(t)$$

and thus a generalisation of the concept of heat capacity may be taken as the ratio between the (complex) surface heat flux and the (complex) time derivative of the surface temperature, yielding:

$$C_d(\omega)=\varepsilon\frac{\tanh(\sqrt{\frac{i\omega}{\alpha}}d)}{\sqrt{i\omega}} \tag{5}$$

where $\varepsilon=\sqrt{\rho ck}$ is the effusivity (see [5] for a definition), and $C_d(\omega)$ is a complex function of ω, sometime termed "complex heat capacity" [6,7]. This definition can then be extended to the periodic (non-harmonic) heating of the 1D-slab, taking the ratio of the Fourier transforms of $\dot{u}(t)$ and $\dot{T}_0(t)$, obtaining exactly the same result (5).

Generally:

$$\dot{u}_d(t)=q_0(t)=\int_{-\infty}^{+\infty}C_d(\omega)i\omega S_0(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

and from the convolution theorem [9]:

$$\dot{u}_d(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}K_d(t-s)\dot{T}_0(s)\,\mathrm{d}s \tag{6}$$

with: $K_d(t)=\int_{-\infty}^{+\infty}C_d(\omega)e^{i\omega t}\,\mathrm{d}\omega$ representing the dynamic heat capacity in time domain. The definitions are clearly referred to a unit of surface extension (1 m^2) and in the limiting case of $\omega\to 0$ the dynamic heat capacity in frequency and time domain take the reasonable forms:

$$C_d(\omega)=\varepsilon\frac{\tanh(\sqrt{\frac{i\omega}{\alpha}}d)}{\sqrt{i\omega}}\to\rho cd$$
$$K_d(t)\to 2\pi\rho cd\,\delta(t)$$

consistent with the classical definition of "static" heat capacity. It is interesting to observe that the dynamic heat capacity of a semi-infinite slab is then:

$$C_\infty(\omega)=\varepsilon\frac{1}{\sqrt{i\omega}}$$

and

$$K_\infty(t)=\varepsilon\int_{-\infty}^{+\infty}\frac{1}{\sqrt{i\omega}}e^{i\omega t}\,\mathrm{d}\omega=\begin{cases}\varepsilon\frac{2\sqrt{\pi}}{|t|^{1/2}} & \text{for } t>0\\ 0 & \text{for } t<0\end{cases} \tag{7}$$

(see appendix). It is also clear that the instantaneous energy content of a slab (finite or semi-infinite) can be obtained, disregarding the time-averaged value, through the use of K_d (or K_∞) as:

$$u_d(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}K_d(t-s)T_0(s)\,\mathrm{d}s \tag{8}$$

$$u_\infty(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}K_\infty(t-s)T_0(s)\,\mathrm{d}s \tag{9}$$

4. The dependence of the storage capacity on the heating profile

The heat storage capacity of the 1D slab can be defined as follows. Consider the total internal energy fluctuation defined as:

$$u(t)=U(t)-U_a$$

where $U_a=\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}U(t)\,\mathrm{d}t$ ($\mathcal{T}$ is the period) is the time-average energy content of the slab, the difference between the maximum and the minimum values of $u(t)$ is then defined as the storage capacity. Eqs. (3), (4) and (8), (9) allow to evaluate the instantaneous energy content of the slab and can be used alternatively, as the first ones work in the frequency domain while the others work in time domain, and for different cases one can be better used than the other. The case of semi-infinite slab will be firstly considered; as it will become evident later, the problem of the slab of finite thickness can be naturally connected to the results for the semi-infinite one.

4.1. The semi-infinite slab

Consider the reference case of harmonic heating (with surface temperature fluctuation amplitude equal to T_0), the instantaneous energy content is then:

$$\begin{aligned}u_\infty(t)&=U_\infty(t)-U_{\infty,a}\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}K_\infty(t-s)T_0(s)\,\mathrm{d}s\\&=2\varepsilon\sqrt{\pi}\frac{1}{2\pi}\operatorname{Re}\left\{\int_{-\infty}^{t}\frac{1}{|t-s|^{1/2}}T_0e^{i\omega s}\,\mathrm{d}s\right\}\\&=\frac{\varepsilon}{\sqrt{2}}\frac{T_0}{\sqrt{|\omega|}}\big(\cos(\omega t)+\sin(\omega t)\big)\end{aligned}$$

The maximum of $U_\infty(t)$ is obtained for $t\omega=\frac{\pi}{4}$ and the minimum for $t\omega=\frac{5\pi}{4}$, then

$$\max\{U_\infty(t)\}=\varepsilon\frac{T_0}{\sqrt{|\omega|}};\qquad \min\{U_\infty(t)\}=-\varepsilon\frac{T_0}{\sqrt{|\omega|}}$$

and the heat storage capacity is $Q_{h,\infty}=[\max\{U_\infty(t)\}-\min\{U_\infty(t)\}]=\frac{2\varepsilon}{\sqrt{\omega}}T_0=\frac{\sqrt{2}\varepsilon\sqrt{\mathcal{T}}}{\sqrt{\pi}}T_0$, see also [4].

As an example of the use of the equation in time domain it is interesting to evaluate the storage capacity of a semi-infinite slab subject to an intermittent heating defined as:

$$T_0(t)=\begin{cases}2T_0 & \text{for } 0<t<t_0\\ 0 & \text{for } t_0<t<\mathcal{T}\end{cases} \tag{10}$$

noticing that the temperature excursion and period are equal to those of the reference (harmonic) case. The average surface temperature is then:

$$T_a=2T_0\frac{t_0}{\mathcal{T}}=2T_0\tau_0$$

Starting again from Eq. (9) and using (7) with $0<t<\mathcal{T}$, the instantaneous energy content of the semi-infinite slab is

$$
\begin{aligned}
u_\infty(t) &= U_\infty(t) - U_a \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} K_\infty(t-s) T_0'(s)\,\mathrm{d}s \\
&= \frac{\varepsilon}{\sqrt{\pi}}\int_{-\infty}^{t} \frac{1}{|t-s|^{1/2}} T_0'(s)\,\mathrm{d}s \\
&= \frac{\varepsilon}{\sqrt{\pi}}\sum_{k=0}^{\infty}\int_{-(k+1)\mathcal{T}}^{-k\mathcal{T}} \frac{1}{|t-s|^{1/2}} T_0'(s)\,\mathrm{d}s \\
&\quad + \frac{\varepsilon}{\sqrt{\pi}}\int_{0}^{t} \frac{1}{|t-s|^{1/2}} T_0'(s)\,\mathrm{d}s
\end{aligned}
$$

where $T_0'(t) = T_0(t) - T_a$. After some manipulations the first integral becomes:

$$
\begin{aligned}
r(t) &= \sum_{k=0}^{\infty}\int_{-(k+1)\mathcal{T}}^{-k\mathcal{T}} \frac{1}{|t-s|^{1/2}} T_0'(s)\,\mathrm{d}s \\
&= 2T_0\sum_{k=0}^{\infty}\bigl\{2\bigl[\sqrt{t+(k+1)\mathcal{T}} - \sqrt{t-t_0+(k+1)\mathcal{T}}\,\bigr] \\
&\quad - 2\tau_0\bigl[\sqrt{t+(k+1)\mathcal{T}} - \sqrt{t+k\mathcal{T}}\,\bigr]\bigr\} \\
&= 4T_0\sqrt{\mathcal{T}}\biggl\{\zeta\left(-\frac{1}{2}, \tau+1\right) \\
&\quad - \zeta\left(-\frac{1}{2}, \tau+1-\tau_0\right) + \tau_0\sqrt{\tau}\biggr\}
\end{aligned}
$$

with $\tau = t/\mathcal{T}$ and $\zeta(s,a)$ is the Hurwitz zeta function [8]. The second integral is instead:

$$
\begin{aligned}
h(t) &= \int_{0}^{t} \frac{1}{|t-s|^{1/2}} T_0'(s)\,\mathrm{d}s \\
&= \begin{cases} 4T_0\sqrt{\mathcal{T}}(1-\tau_0)\sqrt{\tau} & \text{for } \tau < \tau_0 \\ 4T_0\sqrt{\mathcal{T}}[\sqrt{\tau}(1-\tau_0) - \sqrt{\tau-\tau_0}] & \text{for } \tau > \tau_0 \end{cases}
\end{aligned}
$$

and $u_\infty(t) = \frac{\varepsilon}{\sqrt{\pi}}\{r(t)+h(t)\}$. Fig. 1 shows a computation of $U(t) - U_{\min}$ for different values of the parameter τ_0 and it can be noticed that the amplitude fluctuation depends on τ_0 and it may become larger that the corresponding fluctuation for the harmonic case. It is easy to show that $r'(t = t_0) < -h'(t = t_0) = +\infty$, and then the maximum of U_∞ is located at $t = t_0$ whereas the minimum is located at $t = 0$, then

$$
\begin{aligned}
Q_{i,\infty} &= U_{\max} - U_{\min} \\
&= \frac{\varepsilon}{\sqrt{\pi}}\bigl\{r(t_0) - r(0) + h(t_0) - h(0)\bigr\} \\
&= 2\sqrt{2}\biggl\{\zeta\left(-\frac{1}{2}, \tau_0+1\right) + \zeta\left(-\frac{1}{2}, 1-\tau_0\right) \\
&\quad - 2\zeta\left(-\frac{1}{2}, 1\right) + \sqrt{\tau_0}\biggr\} Q_{h\infty} \\
&= \Psi(\tau_0)\ Q_{h\infty}
\end{aligned}
$$

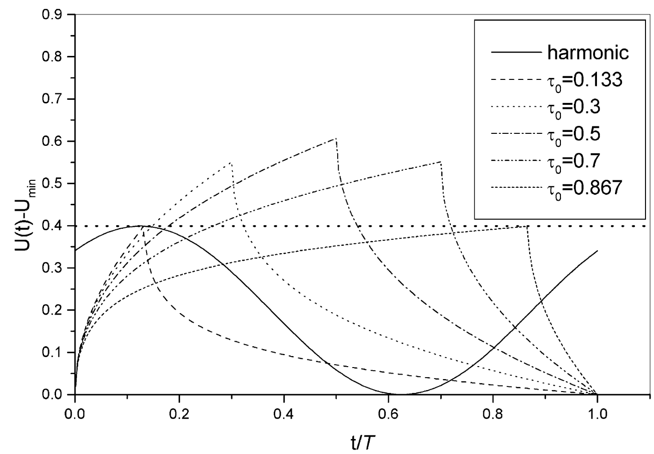

Fig. 1. Instantaneous internal energy content in a semi-infinite slab for intermittent (with different values of τ_0) and harmonic heating.

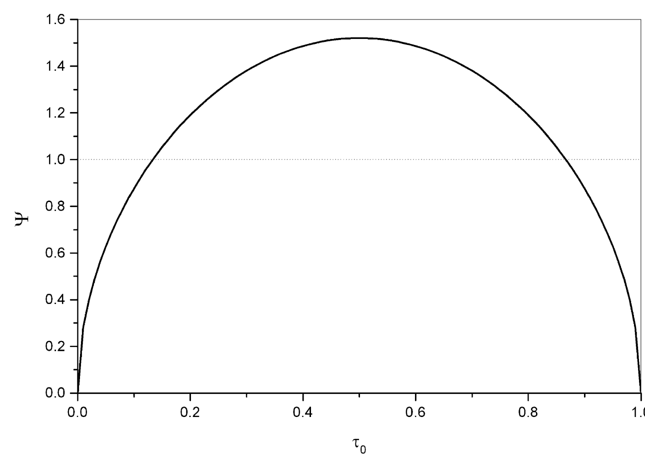

Fig. 2. Ratio between the heat storage capacity of a semi-infinite slab under intermittent heating and that of the same slab harmonically heated ($\Psi = \frac{Q_{i,\infty}}{Q_{h,\infty}}$).

Fig. 2 shows the numerical evaluation of the function $\Psi(\tau_0)$. It is clear that the storage capacity is larger than that obtained by harmonic heating, as long as $0.133 \lesssim \tau_0 \lesssim 0.867$, and for $\tau_0 = 0.5$ the storage capacity reaches its maximum value equal to 1.52 times that of a harmonically heated (with the same amplitude and period) semi-infinite slab. This comparison shows that the efficiency of a semi-infinite slab as heat storage depends on the heating time profile and is not optimised for harmonic input.

4.2. The slab of finite thickness

Consider now a slab of finite thickness d, then the harmonic heating, that as above mentioned was treated in detail in [4], can be analysed by using the equation in the frequency domain (3), by setting

$$S_0(\omega) = T_0\delta(\omega - \omega_0)$$

The instantaneous energy content is then:

$$u_d(t) = \rho c d\,\mathrm{Re}\left\{\int_{-\infty}^{+\infty} S_0(\omega)\frac{\tanh(\beta d)}{\beta d}e^{i\omega t}\,\mathrm{d}\omega\right\}$$

$$= \frac{\varepsilon T_0}{\sqrt{2}\sqrt{\omega}}\left[\left\{\frac{\{\sinh(2\xi)+\sin(2\xi)\}}{\cosh(2\xi)+\cos(2\xi)}\right\}\cos(\omega t) - \left\{\frac{\{\sin(2\xi)-\sinh(2\xi)\}}{\cosh(2\xi)+\cos(2\xi)}\right\}\sin(\omega t)\right]$$

with $\xi = d\sqrt{\frac{\omega}{2\alpha}}$, the relative maximum and minimum are found for:

$$\tan(\omega t_{1,2}) = \frac{\sinh(2\xi)-\sin(2\xi)}{\sinh(2\xi)+\sin(2\xi)}$$

$$\omega t_1 = -\arctan\left(\frac{\sin(2\xi)}{\sinh(2\xi)}\right) - \frac{3}{4}\pi$$

$$\omega t_2 = -\arctan\left(\frac{\sin(2\xi)}{\sinh(2\xi)}\right) - \frac{1}{4}\pi$$

and after some manipulation the heat storage can be written in the more convenient form:

$$Q_{h,d} = u_d(t_1) - u_d(t_2) = \frac{2\varepsilon T_0\sqrt{(\cosh(2\xi)-\cos(2\xi))}}{\sqrt{\omega}\sqrt{\{\cosh(2\xi)+\cos(2\xi)\}}} = \frac{\sqrt{(\cosh(2\xi)-\cos(2\xi))}}{\sqrt{\cosh(2\xi)+\cos(2\xi)}}Q_{h,\infty}$$

Magyari and Keller [4] have shown that the first relative (and absolute) maximum of the function $\Lambda_h(\xi) = \frac{\sqrt{(\cosh(2\xi)-\cos(2\xi))}}{\sqrt{\cosh(2\xi)+\cos(2\xi)}}$ is reached at $\xi = 1.18251$ and its value is 1.14299.

Consider now the case of the generally periodic heating. The function K_d can be written in a more convenient form using the convolution theorem [9]:

$$K_d(t) = \varepsilon\int_{-\infty}^{+\infty}\frac{\tanh(\beta d)}{\sqrt{i\omega}}e^{i\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}K_\infty(t-s)k_d(s)\,\mathrm{d}s$$

where

$$k_d(t) = \int_{-\infty}^{+\infty}\tanh(\beta d)e^{i\omega t}\,\mathrm{d}\omega; \qquad K_\infty(t) = \varepsilon\int_{-\infty}^{+\infty}\frac{1}{\sqrt{i\omega}}e^{i\omega t}\,\mathrm{d}\omega$$

The instantaneous energy content of the slab can be written as:

$$u_d(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}K_d(t-s)T_0(s)\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}K_\infty(t-s-p)T_0(s)\,\mathrm{d}s\right]k_d(p)\,\mathrm{d}p = \frac{1}{2\pi}\int_{-\infty}^{+\infty}u_\infty(s)k_d(t-s)\,\mathrm{d}s$$

and the function $k_d(t) = \int_{-\infty}^{+\infty}\tanh(\beta d)e^{i\omega t}\,\mathrm{d}\omega$, which is independent of the particular heating profile, allows to relate the results obtained for the semi-infinite slab to that of the finite one (under the same heating input). The properties of the function $k_d(t)$ are better analysed in Appendix B, and from them it is possible to evaluate the slab instantaneous energy content as:

$$u_d(t) = \frac{1}{2\pi}\int_0^\infty u_\infty\left((\chi-\mu)\frac{d^2}{2\alpha}\right)\kappa_d^*(\mu)\,\mathrm{d}\mu + u_\infty(t)$$

with $\chi = t\frac{2\alpha}{d^2}$. A further simplification can be obtained by using the periodicity of the function $U_\infty(t)$:

$$u_\infty(t) = u_\infty(t+mT) = u_\infty(\sigma + m\mathcal{T}^*) \quad \text{for } m \text{ integer}$$

with $\mathcal{T}^* = \mathcal{T}\frac{2\alpha}{d^2}$; defining:

$$\hat{\kappa}_T(\mu) = \sum_{k=0}^{\infty}\kappa_d^*\big(\mu + (k+1)\mathcal{T}^*\big)$$

it is easy to show that

$$\begin{aligned} u_d(t) &= \sum_{k=0}^{\infty}\frac{1}{2\pi}\int_0^{\mathcal{T}^*}u_\infty\big(p-(k+1)\mathcal{T}^*\big) \times \kappa_d^*\big(\chi - p + (k+1)\mathcal{T}^*\big)\,\mathrm{d}p \\ &\quad + \frac{1}{2\pi}\int_0^{\chi}u_\infty\left(\sigma\frac{d^2}{2\alpha}\right)\kappa_d^*(\chi-\sigma)\,\mathrm{d}\sigma + u_\infty(t) \\ &= \frac{1}{2\pi}\int_0^{T^*}u_\infty(\sigma)\big\{\widehat{\kappa}_{\mathcal{T}}(\chi-\sigma) + H(\chi-\sigma)\kappa_d^*(\chi-\sigma)\big\}\,\mathrm{d}\sigma + u_\infty(t) \\ &= \frac{1}{2\pi}\int_0^{T^*}u_\infty(\sigma)\widehat{K}_{\mathcal{T}}(\chi-\sigma)\,\mathrm{d}\sigma + u_\infty(t) \end{aligned} \qquad (11)$$

where:

$$H(\sigma) = \begin{cases} 1 & \text{for } \sigma > 0 \\ 0 & \text{for } \sigma < 0 \end{cases}$$

$$\widehat{K}_{\mathcal{T}}(\chi-\sigma) = \widehat{\kappa}_{\mathcal{T}}(\chi-\sigma) + H(\chi-\sigma)\kappa_d^*(\chi-\sigma)$$

The function $\kappa_d^*(\sigma)$ is independent of the period $\mathcal{T}$, whereas $\widehat{K}_{\mathcal{T}}(\sigma)$ depends explicitly on $\mathcal{T}$. Eq. (11) allows, once that $\widehat{K}_{\mathcal{T}}(\mu)$ has been numerically evaluated, to calculate the instantaneous energy content for any temperature fluctuation profile with period $\mathcal{T}$. As a test, the harmonic heating case was considered, using Eq. (11) the heat storage was calculated and the results show an agreement with the analytical solution (see Fig. 3) which is only limited by the chosen integration step.

Eq. (11) was then applied to the intermittent heating case (Eq. (10)), by analogy with the harmonic case, the non-dimensional slab thickness was defined as: $\xi = \mathrm{d}\sqrt{\frac{2\pi}{2\alpha\mathcal{T}}}$. The ratio between the heat storage capacity of a slab of finite thickness d and the corresponding heat storage of infinite slab subject to the intermittent heating ($\Lambda_i(\xi) = \frac{Q_{i,d}}{Q_{i,\infty}}$) is reported in Fig. 4 for different values of the parameter τ_0. The maximum capacity is always larger than that of the semi-infinite medium,

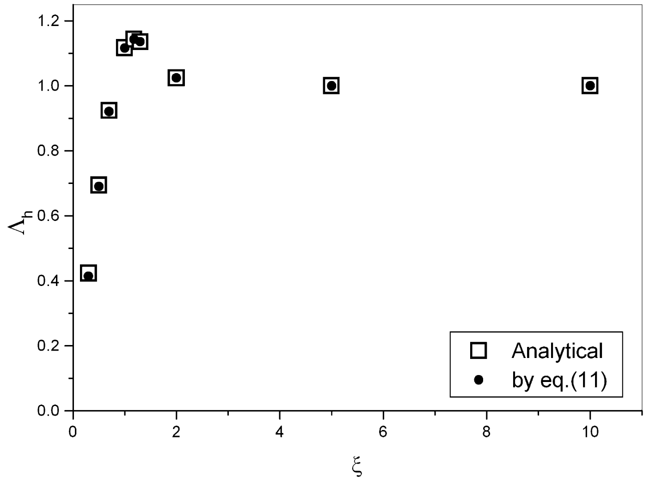

Fig. 3. Ratio between the heat storage capacity of a finite slab and that of the semi-infinite slab under harmonic heating, evaluated through Eq. (11) and analytically by $\Lambda_h = \frac{Q_{h,d}}{Q_{h,\infty}} = \frac{\sqrt{(\cosh(2\xi)-\cos(2\xi))}}{\sqrt{\cosh(2\xi)+\cos(2\xi)}}$.

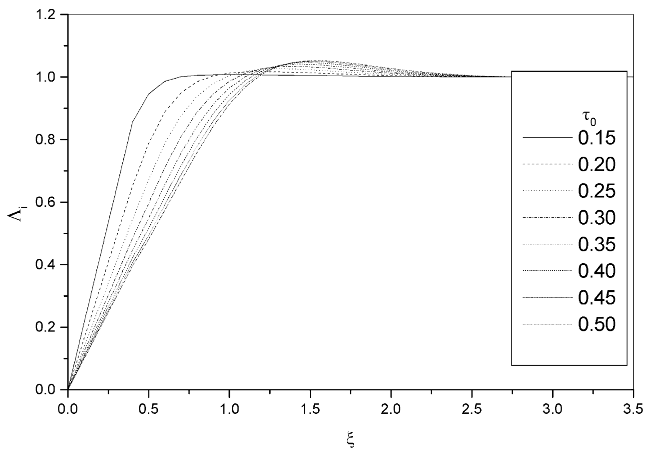

Fig. 4. Ratio between the heat storage capacity of a finite thickness slab and that of a semi-infinite one under intermittent heating ($\Lambda_i = \frac{Q_{i,d}}{Q_{i,\infty}}$).

but the maximum is reached for different values of the thickness and reaches different values. Fig. 5 shows the values of the "optimum" non-dimensional slab thickness as a function of τ_0. It is also interesting to consider the ratio between the maximum heat storage capacity reached with intermittent heating (for different τ_0), and the maximum reached by harmonic heating (with the same period and amplitude). Fig. 6 shows the results and it should be noticed that the maximum of the heat storage is referred to the optimum slab thickness which is different for each excitation profile. The intermittent heating becomes better than the harmonic one only when $0.81 \gtrsim \tau_0 \gtrsim 0.19$, and for $\tau_0 = 0.5$, the maximum heat storage capacity for the intermittent heating is 40% larger than that obtained by harmonic heating. As a final remark, it may be noticed that for very high frequencies, possible non-Fourier effects may become important and in such case the hyperbolic version of the heat equation can be used (as in [4], see also [10]), although the expected very low value of the relaxation time for common materials may fade the interest of this case for practical applications.

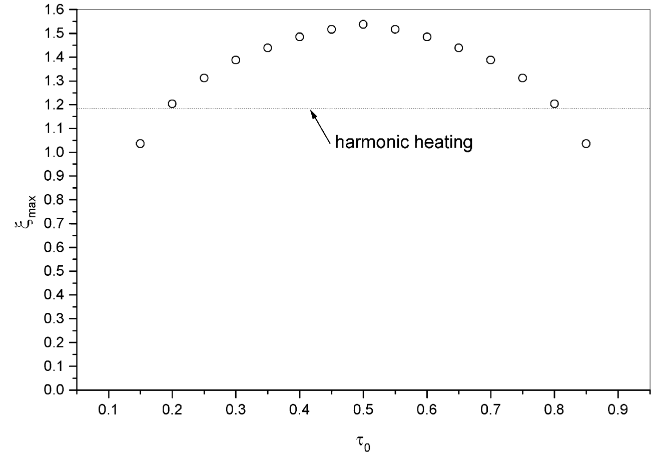

Fig. 5. Non-dimensional slab thickness for maximum heat storage capacity under intermittent heating.

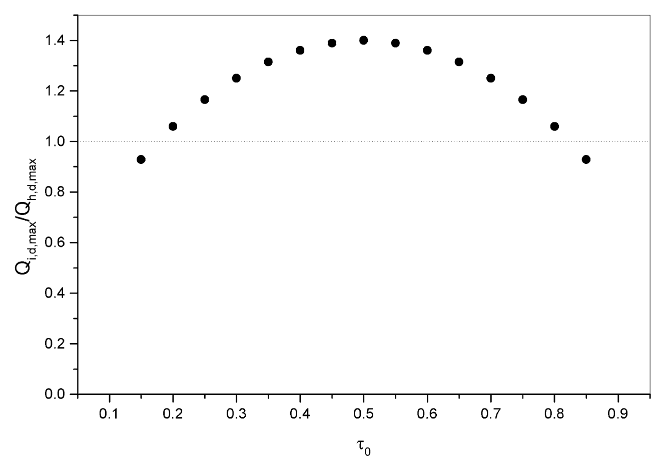

Fig. 6. Ratio between the maximum heat storage capacity of a finite thickness slab under intermittent heating and that of a slab harmonically heated.

5. Conclusions

The general case of periodic (not necessarily harmonic) heating of finite and semi-infinite slab was analysed in terms of heating storage. Equations to calculate the heat storage in time and frequency domain were obtained. For a slab of finite thickness, the instantaneous energy content was linearly related to the analogous content for a semi-infinite slab. As for the harmonic heating, an optimum slab thickness exists that maximises the heat storage capacity and it depends on the heating profile and the heating period. It was found that the harmonic heating is not the optimum one, in terms of heat storage, as intermittent heating (with the same heating period and amplitude) may yield a maximum heat storage up to 1.40 times (and up to 1.52 times for the semi-infinite slab) that obtained by harmonic heating . Moreover, for intermittent heating, an analytic dependence of the heat storage on the shape parameter τ_0 was fund. The analysed case of intermittent heating then revealed that the optimum heat storage and slab thickness depend on the heating profile.

Appendix A

In time domain, the dynamic heat capacity of a semi-infinite slab is represented by the function:

$$K_\infty(t) = \varepsilon \int_{-\infty}^{+\infty} \frac{1}{\sqrt{i\omega}} e^{i\omega t}\, \mathrm{d}\omega$$

noticing that:

$$\sqrt{i\omega} = \begin{cases} \frac{(1+i)}{\sqrt{2}}\sqrt{|\omega|} & \text{for } \omega > 0 \\ \frac{(1-i)}{\sqrt{2}}\sqrt{|\omega|} & \text{for } \omega < 0 \end{cases}$$

the integral becomes:

$$g(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{i\omega}} e^{i\omega t}\, \mathrm{d}\omega$$

$$= \frac{\sqrt{2}}{(1-i)} \int_{-\infty}^{0} \frac{1}{\sqrt{|\omega|}} e^{i\omega t}\, \mathrm{d}\omega + \frac{\sqrt{2}}{(1+i)} \int_{0}^{+\infty} \frac{1}{\sqrt{|\omega|}} e^{i\omega t}\, \mathrm{d}\omega$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{-\infty} \frac{1}{\sqrt{|\omega|}} e^{i\omega t}\, \mathrm{d}\omega + \frac{2}{\sqrt{2}} \left[\int_{0}^{\infty} \frac{1}{\sqrt{|\omega|}} \sin(\omega t)\, \mathrm{d}\omega \right]$$

Remembering that (see [9])

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{|\omega|}} e^{i\omega t}\, \mathrm{d}\omega = \frac{\sqrt{2\pi}}{2} \frac{1}{|t|^{1/2}}$$

$$\int_{0}^{\infty} \frac{1}{\sqrt{|\omega|}} \sin(\omega t)\, \mathrm{d}\omega = \frac{\sqrt{2\pi}}{2} \frac{t}{|t|^{3/2}}$$

the following result stems:

$$K_\infty(t) = \varepsilon \int_{-\infty}^{+\infty} \frac{1}{\sqrt{i\omega}} e^{i\omega t}\, \mathrm{d}\omega = \begin{cases} \varepsilon \frac{2\sqrt{\pi}}{|t|^{1/2}} & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Appendix B

Introducing the non-dimensional variables: $\eta = \frac{\omega d^2}{2\alpha} = \xi^2$ and $\sigma = \frac{2\alpha t}{d^2}$ and defining the non-dimensional functions:

$$\kappa_d^{e*}(\sigma) = 2 \int_0^\infty \frac{\sinh(2\sqrt{\eta}) - \cosh(2\sqrt{\eta}) - \cos(2\sqrt{\eta})}{[\cosh(2\sqrt{\eta}) + \cos(2\sqrt{\eta})]} \cos(\eta\sigma)\, \mathrm{d}\eta$$

$$\kappa_d^{o*}(\sigma) = -2 \int_0^\infty \frac{\sin(2\sqrt{\eta})}{[\cosh(2\sqrt{\eta}) + \cos(2\sqrt{\eta})]} \sin(\eta\sigma)\, \mathrm{d}\eta$$

with:

$$\kappa_d^{e*}(-\sigma) = \kappa_d^{e*}(\sigma); \qquad \kappa_d^{o*}(-\sigma) = -\kappa_d^{o*}(\sigma)$$

and noticing that:

$$2 \int_0^\infty \frac{\sinh(2\sqrt{\eta}) \cos(\eta\sigma)}{[\cosh(2\sqrt{\eta}) + \cos(2\sqrt{\eta})]}\, \mathrm{d}\eta$$

$$= 2 \int_0^\infty \frac{\sinh(2\sqrt{\eta}) - \cosh(2\sqrt{\eta}) - \cos(2\sqrt{\eta})}{[\cosh(2\sqrt{\eta}) + \cos(2\sqrt{\eta})]} \cos(\eta\sigma)\, \mathrm{d}\eta$$

$$+ 2\pi\delta(\sigma) = \kappa_d^{e*}(\sigma) + 2\pi\delta(\sigma)$$

the function $k_d(t)$ can be written as:

$$k_d = \frac{2\alpha}{d^2} \kappa_d(\sigma)$$

$$= \frac{2\alpha}{d^2} \int_0^\infty 2 \frac{\sinh(2\sqrt{\eta}) \cos(\eta\sigma) - \sin(2\sqrt{\eta}) \sin(\eta\sigma)}{[\cosh(2\sqrt{\eta}) + \cos(2\sqrt{\eta})]}\, \mathrm{d}\eta$$

$$= \frac{2\alpha}{d^2} \left\{ \kappa_d^{e*}(\sigma) + \kappa_d^{o*}(\sigma) + 2\pi\delta(\sigma) \right\}$$

From the observation that for $\sigma > 0$: $\kappa_d^{e*}(\sigma) = \kappa_d^{o*}(\sigma)$ we have:

$$\kappa_d(\sigma) = \kappa_d^{*}(\sigma) + 2\pi\delta(\sigma) = \kappa_d^{e*}(\sigma) + \kappa_d^{o*}(\sigma) + 2\pi\delta(\sigma)$$

$$\kappa_d^{*}(\sigma) = \begin{cases} 2\kappa_d^{e*}(\sigma) & \text{for } \sigma > 0 \\ 0 & \text{for } \sigma < 0 \end{cases}$$

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